Practice Final Exam I

We strongly recommend that you work through this exam under realistic conditions rather than just flipping through the problems and seeing what they look like. Setting aside three hours in a quiet space with your notes and making a good honest effort to solve all the problems is one of the single best things you can do to prepare for this exam. It will give you practice working under time pressure and give you an honest sense of where you stand and what you need to get some more practice with.

This practice final exam is essentially the final exam from Winter 2018. The sorts of questions here are representative of what you might expect to get on the upcoming final exam, though the point balance and distribution of problems might be a bit different.

The exam is closed-book, closed-computer, limited note (one double-sided sheet of 8.5” × 11” paper decorated however you'd like).

You have three hours to complete this exam. There are 71 total points.

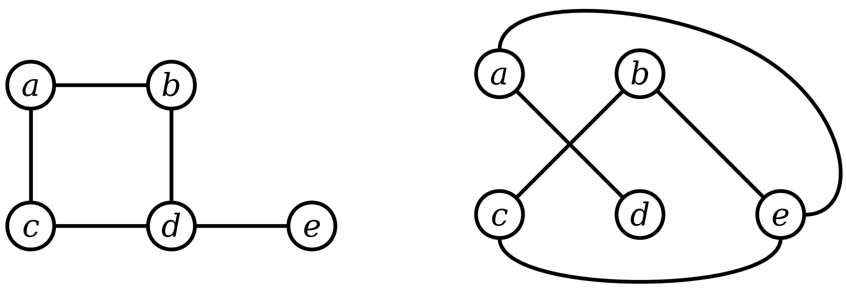
Question	**Points**
(1) Graphs and Pigeonhole	/ 10
(2) Equivalence Relation, Functions, and Sets	/ 19
(3) Strict Orders and Induction	/ 18
(4) Regular Languages	/ 14
(5) **R** and **RE** Languages	/ 10
	/ 71

Problem One: Graphs and the Pigeonhole Principle (10 Points)

*(We recommend spending about **40** minutes on this problem.)*

On Problem Set Four, you explored bipartite graphs. As a refresher, a graph $G = (V, E)$ is called ***bipartite*** if its nodes can be partitioned into two sets V_1 and V_2 where every edge $e \in E$ has one endpoint in V_1 and the other in V_2. This question explores some additional properties of bipartite graphs.

Let's begin with a new definition. If $G = (V, E)$ is an undirected graph, the ***complement of G***, denoted $\boldsymbol{G^c}$, is a graph related to the original graph G. Intuitively, G^c has the same nodes as G, and its edges consist of all the edges missing from graph G. Formally speaking, G^c is the graph with the same nodes as G and with edges determined as follows: the edge $\{u, v\}$ is present in G^c if and only if $u \neq v$ and the edge $\{u, v\}$ is not present in G. As an example, here's a sample graph and its complement:



Prove that if G is a graph with at least five nodes, then at least one of G and G^c is ***not*** bipartite.

Problem Two: Equivalence Relations, Functions, and Sets (19 Points)

(We recommend spending about 40 minutes on this problem.)

Equivalence relations are a workhorse in discrete mathematics and can be used to rigorously pin down all sorts of structures. This problem explores an important operation on equivalence relations and its properties.

Let's begin with a refresher on a definition. If R is an equivalence relation over a set A and $x \in A$, then the ***equivalence class of x*** with respect to R, denoted $\mathbf{[x]_R}$, is the set

$$[x]_R \;=\; \{\, y \in A \mid xRy \,\}.$$

Intuitively, $[x]_R$ consists of all the elements of A that x is related to by R.

Now, a new definition. If R is an equivalence relation over a set A, then the set $\boldsymbol{A / R}$ is the set of all the equivalence classes of the elements of A. Formally speaking, we say that

$$A \,/\, R \;=\; \{\, [x]_R \mid x \in A \,\}.$$

This set is sometimes called the ***quotient set*** of R.

i. **(3 Points)** Consider the following binary relation E over the set $\mathbb{N}^2$:

$$(m_1, n_1)\ E\ (m_2, n_2) \quad \text{if} \quad m_1 + m_2 \text{ is even and } n_1 + n_2 \text{ is even.}$$

(Remember that this "if" means "is defined as" and is not an implication.)

What is $|\mathbb{N}^2 \,/\, E|$? Briefly justify your answer, but no formal proof is required.

Here's a terminology refresher from the previous page:

$$[x]_R \;=\; \{\, y \in A \mid xRy \,\} \qquad\qquad A \,/\, R \;=\; \{\, [x]_R \mid x \in A \,\}.$$

Although equivalence relations come in all sorts of shapes and flavors, there is a single equivalence relation that's, in some sense, the "most fundamental" equivalence relation: the equality relation. In the remainder of this problem, you'll show that every equivalence relation's behavior can be thought of as the behavior of the equals relation over some well-chosen collection of sets.

Let R be an equivalence relation over a set A. We can define a function $f : A \to A \,/\, R$ as follows:

$$f(x) = [x]_R.$$

That is, f maps each element of A to its equivalence class.

ii. **(3 Points)** Below are a series of statements about the behavior of this function f. For each statement, decide whether that statement is always true regardless of what R is, always false regardless of what R is, or whether it depends on the choice of R. (Remember that R is assumed to be an equivalence relation.) No justification is necessary. There is no penalty for an incorrect guess.

The function f is injective.

☐ True, regardless of what R is. ☐ False, regardless of what R is. ☐ It depends on R.

The function f is surjective.

☐ True, regardless of what R is. ☐ False, regardless of what R is. ☐ It depends on R.

The function f is bijective.

☐ True, regardless of what R is. ☐ False, regardless of what R is. ☐ It depends on R.

As a refresher from the previous page, we've let R be an *equivalence relation* over some set A. We've used the following terminology:

$$[x]_R = \{ y \in A \mid xRy \} \qquad\qquad A / R = \{ [x]_R \mid x \in A \}.$$

We've also let $f : A \to A / R$ be a function defined as $f(x) = [x]_R$.

This function has a wonderful property:

For any $a \in A$ and any $b \in A$, we have aRb if and only if $f(a) = f(b)$.

In the next two parts of this problem, we'd like you to prove this.

iii. **(5 Points)** Using a proof by contrapositive, prove that for any $a, b \in A$ that if $f(a) = f(b)$, then aRb.

Feel free to use this space for scratch work. There's room to write your answer on the next page of this exam.

(Extra space for your answer to Problem Two, Part (iii), if you need it.)

As a refresher from the previous page, we've let R be an ***equivalence relation*** over some set A. We've used the following terminology:

$$[x]_R \;=\; \{\, y \in A \mid xRy \,\} \qquad\qquad A\,/\,R \;=\; \{\, [x]_R \mid x \in A \,\}.$$

We've also let $f : A \to A\,/\,R$ be a function defined as $f(x) = [x]_R$.

iv. **(8 Points)** Prove that if a, $b \in A$ and aRb, then $f(a) = f(b)$. As a hint, how do you show that two sets are equal to one another?

Feel free to use this space for scratch work. There's room to write your answer on the next page of this exam.

(Extra space for your answer to Problem Two, Part (iv), if you need it.)

Problem Three: Strict Orders and Induction (18 Points)

*(We recommend spending about **35** minutes on this problem.)*

Let's begin with a new definition. If R and S are binary relations over the same set A, then the ***composition of R and S***, denoted $\boldsymbol{R \circ S}$, is a binary relation over A defined as follows:

$$x(R \circ S)y \quad \text{if} \quad \exists z \in A.\ (xRz \land zSy).$$

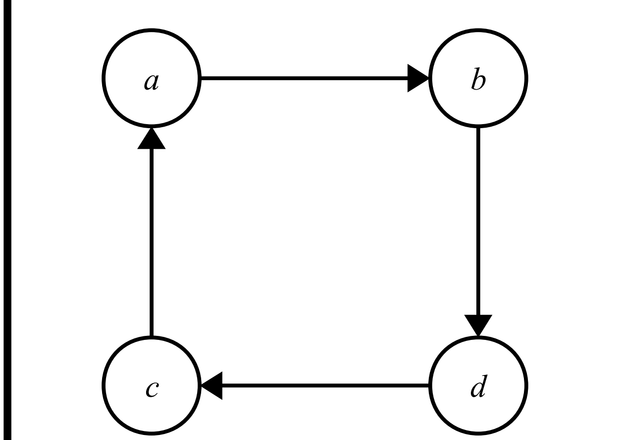
Having defined the composition of two relations, we can inductively define the ***nth power*** of a binary relation R over a set A as follows:

$$xR^1y \quad \text{if} \quad xRy$$

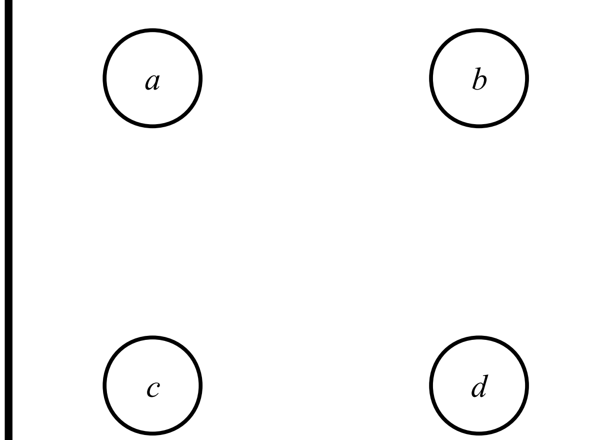
$$xR^{n+1}y \quad \text{if} \quad x(R \circ R^n)y$$

Remember that the word "if" in the above contexts means "is defined as" and is not an implication, and note that R^n is only defined when $n \geq 1$.

i. **(3 Points)** Below is a drawing of a binary relation R over the set $\{a, b, c, d\}$. In the indicated space, draw a picture of R^2. No justification is necessary.



The relation R.



Draw R^2 here.

As refreshers from the previous page, if R and S are binary relations over the same set A, then the relation $\boldsymbol{R \circ S}$ is a binary relation over A defined as follows:

$$x(R \circ S)y \quad \text{if} \quad \exists z \in A.\ (xRz \land zSy).$$

The ***nth power*** of a binary relation R over a set A is defined as follows:

$$\begin{array}{ll} xR^1y & \text{if} \quad xRy \\ xR^{n+1}y & \text{if} \quad x(R \circ R^n)y \end{array}$$

Remember that the word "if" in the above contexts means "is defined as" and is not an implication, and note that R^n is only defined when $n \geq 1$.

ii. **(15 Points)** Consider the $<$ relation over the set $\mathbb{R}$. Prove that the following is true for all nonzero natural numbers n:

$$\forall x \in \mathbb{R}.\ \forall y \in \mathbb{R}.\ (x < y \quad \leftrightarrow \quad x <^n y).$$

Feel free to use the inequality $a < \frac{a+b}{2} < b$, which is true for any real numbers a and b where $a < b$. You can assume that the $<$ relation over $\mathbb{R}$ is a strict order.

(Extra space for your answer to Problem Three, Part (ii), if you need it.)

Problem Four: Regular and Languages (14 Points)

*(We recommend spending about **45** minutes on this problem.)*

If you're living in a world where the only legal arithmetical operation is addition, you can write out a bunch of different expressions that evaluate to odd numbers:

- `3`
- `6 + 3`
- `3 + 3 + 3`
- `6 + 3 + 6 + 3 + 6 + 3 + 6`
- `6 + 6 + 6 + 3 + 6 + 6`
- `3 + 3 + 3 + 3 + 3`

Let Σ = { `3`, `6`, `+` } and consider the following language L_1 over Σ:

L_1 = { $w \in \Sigma^*$ | w doesn't use any numbers besides 3 and 6 and evaluates to an odd number. }

All of the strings shown above are in L_1. Here's a sampler of strings that ***aren't*** in L_1:

- `6` *(this is an even number)*
- `6 + 6` *(this is an even number)*
- `63` *(no multidigit numbers)*
- `+ 6` *(syntactically invalid)*
- `6 ++ 6` *(syntactically invalid)*
- ε *(not a valid expression)*

This turns out to be a regular language.

i. **(3 Points)** Design an NFA for L_1. In the space at the bottom of the page, write a brief explanation (at most two sentences) for how your NFA works.

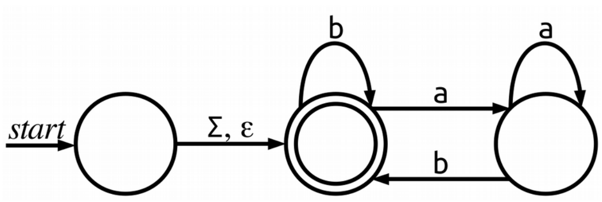
Explanation for this NFA (at most two sentences):

Consider the following regular expression over the alphabet {a, b}:

$$\Sigma?(a^{+}b \;\cup\; b^{*})^{*}$$

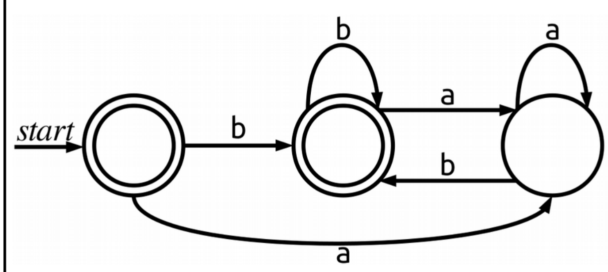
This question explores some properties of this regular expression.

ii. **(4 Points)** Below are four NFAs over the alphabet {a, b}. For each NFA, decide whether the language of that NFA is the same as the language described by the above regular expression. No justification is necessary. There is no penalty for an incorrect guess.



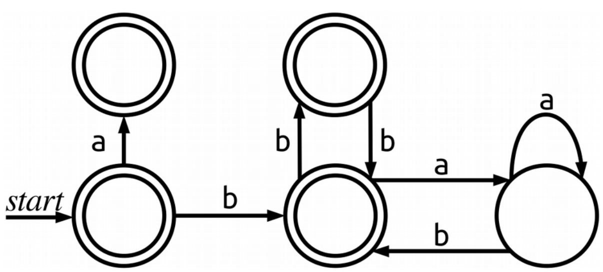
Does this NFA have the same language as the above regular expression?

☐ Yes ☐ No



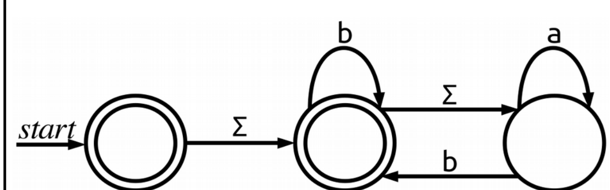
Does this NFA have the same language as the above regular expression?

☐ Yes ☐ No



Does this NFA have the same language as the above regular expression?

☐ Yes ☐ No



Does this NFA have the same language as the above regular expression?

☐ Yes ☐ No

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and consider the finite language $L_2 = \{\varepsilon, \mathtt{a}\}$.

iii. **(7 Points)** Design a DFA for L_2 that is as small as possible. Then, prove that your DFA is as small as possible by using the following theorem that you proved on Problem Set Seven:

> ***Theorem:*** Let L be a language over Σ. Suppose there's a *finite* set S such that any two distinct strings $x, y \in S$ are distinguishable relative to L (that is, $x \not\equiv_L y$ for any two strings $x, y \in S$ where $x \neq y$.) Then any DFA for L must have at least $|S|$ states.

Feel free to use the space below for scratch work. There's room to write your answer on the next page of this exam.

(Extra space for your answer to Problem Four, Part (iii), if you need it.)

Problem Five: R and RE Languages (10 Points)

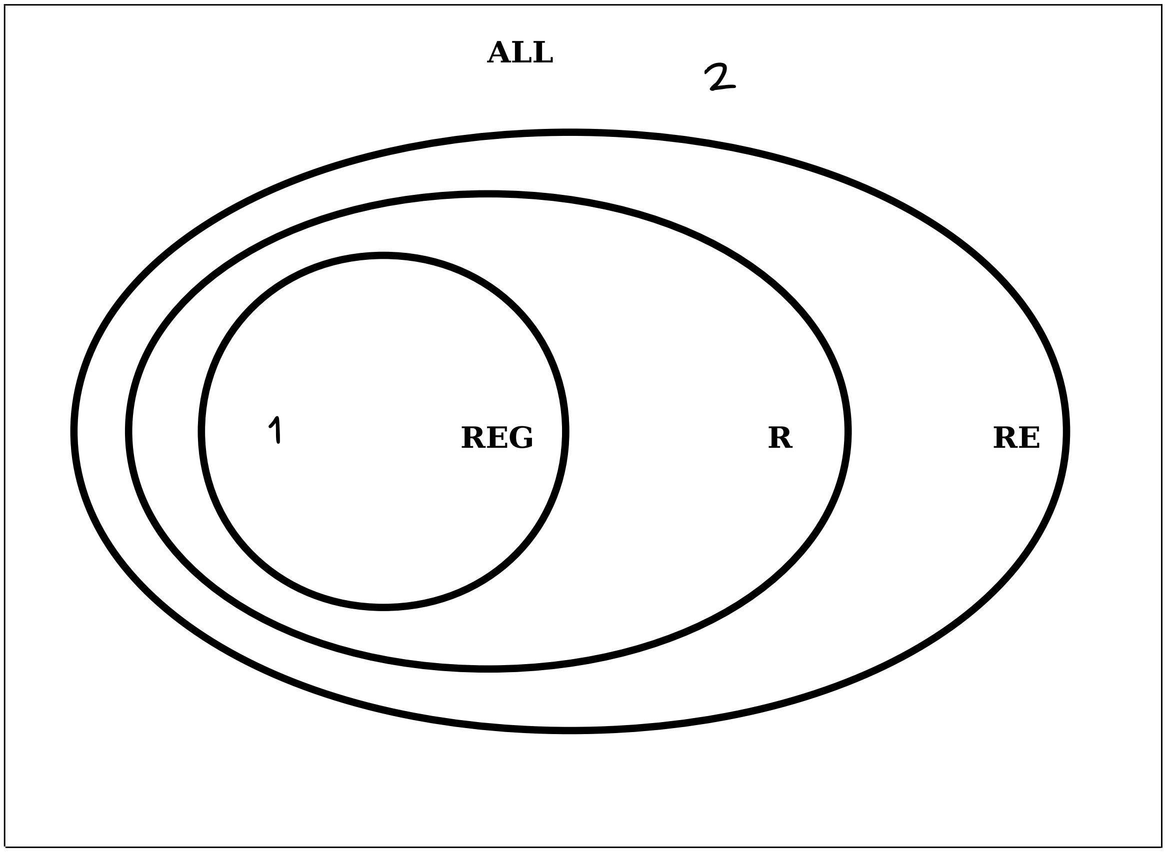
*(We recommend spending about **20** minutes on this problem.)*

Over the past two weeks you've seen your fair share of undecidable problems. If a problem is undecidable, then there's no way to build a TM for that problem that always halts and gives the right answer. However, it's still possible to write a program for an undecidable problem that can get the right answer on many possible inputs – just not *all* of them.

Here are two new definitions. If L is a language over Σ, then a ***sound approximation*** to L is a language S over Σ such that $S \subseteq L$ and a ***complete approximation*** of L is a language C over Σ such that $L \subseteq C$.

i. **(3 Points)** Briefly explain why *every* language has a decidable sound approximation and a decidable complete approximation. This shows that even if a language is undecidable, it's still possible to build a TM that can provide the right answers on some number of strings.

ii. **(7 Points)** Below is a Venn diagram showing the overlap of different classes of languages we've studied so far. We have also provided you a list of numbered languages. For each of those languages, draw where in the Venn diagram that language belongs. As an example, we've indicated where Language 1 and Language 2 should go. No proofs or justifications are necessary, and there is no penalty for an incorrect guess.



1. Σ*
2. L_D
3. { $a^n b^m$ | $n \in \mathbb{N}$ and $m \in \mathbb{N}$ }
4. { ⟨*T*⟩ | *T* is a tournament and the players in *T* are the CS103 staff members }
5. { ⟨*P*⟩ | *P* is a syntactically correct Java program whose source code contains the string `quokka` }
6. { ⟨*P*⟩ | *P* is a syntactically correct Java program that, when run, at some point prints the string `quokka` }
7. { ⟨*P*⟩ | *P* is a syntactically correct Java program that, when run, never prints the string `quokka` }
8. The intersection of languages (5) and (6).
9. { w | $w \in L_D$ }